

A Combinatorial Discussion on Finite Dimensional Leavitt Path Algebras

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Abstract

Any finite dimensional semisimple algebra A over a field K is isomorphic to a direct sum of finite dimensional full matrix rings over suitable division rings. In this paper we will consider the special case where all division rings are exactly the field K . All such finite dimensional semisimple algebras arise as a finite dimensional Leavitt path algebra. For this specific finite dimensional semisimple algebra A over a field K , we define a uniquely determined specific graph - which we name as a truncated tree associated with A - whose Leavitt path algebra is isomorphic to A . We define an algebraic invariant $\kappa(A)$ for A and count the number of isomorphism classes of Leavitt path algebras with $\kappa(A) = n$.

Moreover, we find the maximum and the minimum K -dimensions of the Leavitt path algebras of possible trees with a given number of vertices and determine the number of distinct Leavitt path algebras of a line graph with a given number of vertices.

Keywords: Finite dimensional semisimple algebra, Leavitt path algebra, Truncated trees, Line graphs.

1 Introduction

By the well-known Wedderburn-Artin Theorem [2], any finite dimensional semisimple algebra A over a field K is isomorphic to a direct sum of finite dimensional full matrix rings over suitable division rings. In this paper we will consider the special case where all division rings are exactly the field K . All such finite dimensional semisimple algebras arise as a finite dimensional Leavitt

path algebra as studied in [1]. The Leavitt path algebras are introduced by Abrams and Aranda Pino in 2005, [3]. Many papers on Leavitt path algebras appeared in literature since then. In the following discussion, we are particularly interested in answering some combinatorial questions on the finite dimensional Leavitt path algebras.

We start by recalling the definitions of a path algebra and a Leavitt path algebra, see [1]. A *directed graph* $E = (E^0, E^1, r, s)$ consists of two countable sets E^0, E^1 and functions $r, s : E^1 \rightarrow E^0$. The elements E^0 and E^1 are called *vertices* and *edges*, respectively. For each $e \in E^0$, $s(e)$ is the source of e and $r(e)$ is the range of e . If $s(e) = v$ and $r(e) = w$, then we say that v emits e and that w receives e . A vertex which does not receive any edges is called a *source*, and a vertex which emits no edges is called a *sink*. A graph is called *row-finite* if $s^{-1}(v)$ is a finite set for each vertex v . For a row-finite graph the edge set E^1 of E is finite if its set of vertices E^0 is finite. Thus, a row-finite graph is finite if E^0 is a finite set.

A path in a graph E is a sequence of edges $\mu = e_1 \dots e_n$ such that $r(e_i) = s(e_{i+1})$ for $i = 1, \dots, n-1$. In such a case, $s(\mu) := s(e_1)$ is the *source* of μ and $r(\mu) := r(e_n)$ is the *range* of μ , and n is the *length* of μ , i.e., $l(\mu) = n$.

If $s(\mu) = r(\mu)$ and $s(e_i) \neq s(e_j)$ for every $i \neq j$, then μ is called a *cycle*. If E does not contain any cycles, E is called *acyclic*.

For $n \geq 2$, define E^n to be the set of paths of length n , and $E^* = \bigcup_{n \geq 0} E^n$ the set of all paths.

The path K -algebra over E is defined as the free K -algebra $K[E^0 \cup E^1]$ with the relations:

- (1) $v_i v_j = \delta_{ij} v_i$ for every $v_i, v_j \in E^0$.
- (2) $e_i = e_i r(e_i) = s(e_i) e_i$ for every $e_i \in E^1$.

This algebra is denoted by KE . Given a graph E , define the extended graph of E as the new graph $\widehat{E} = (E^0, E^1 \cup (E^1)^*, r', s')$ where $(E^1)^* = \{e_i^* \mid e_i \in E^1\}$ and the functions r' and s' are defined as

$$r'|_{E^1} = r, \quad s'|_{E^1} = s, \quad r'(e_i^*) = s(e_i) \quad \text{and} \quad s'(e_i^*) = r(e_i).$$

The Leavitt path algebra of E with coefficients in K is defined as the path algebra over the extended graph \widehat{E} , with relations:

$$(CK1) \quad e_i^* e_j = \delta_{ij} r(e_j) \quad \text{for every } e_j \in E^1 \text{ and } e_i^* \in (E^1)^*.$$

$$(CK2) \quad v_i = \sum_{\{e_j \in E^1 \mid s(e_j) = v_i\}} e_j e_j^* \quad \text{for every } v_i \in E^0 \text{ which is not a sink.}$$

This algebra is denoted by $L_K(E)$. The conditions (CK1) and (CK2) are called the Cuntz-Krieger relations. In particular condition (CK2) is the Cuntz-Krieger relation at v_i . If v_i is a sink, we do not have a (CK2) relation at v_i . Note that the condition of row-finiteness is needed in order to define the equation (CK2).

The main structure theorem in [1] can be summarized as follows:

For any $v \in E^0$, we define $n(v) = |\{\alpha \in E^* \mid r(\alpha) = v\}|$.

Proposition 1 :

1. *The Leavitt path algebra $L_K(E)$ is a finite-dimensional K -algebra if and only if E is a finite and acyclic graph.*
2. *If $A = \bigoplus_{i=1}^s M_{n_i}(K)$, then $A \cong L_K(E)$ for a graph E having s connected components each of which is an oriented line graph with n_i vertices, $i = 1, 2, \dots, s$.*
3. *A finite dimensional K -algebra A arises as a $L_K(E)$ for a graph E if and only if $A = \bigoplus_{i=1}^s M_{n_i}(K)$.*
4. *If $A = \bigoplus_{i=1}^s M_{n_i}(K)$ and $A \cong L_K(E)$ for a finite, acyclic graph E , then the number of sinks of E is equal to s , and each sink v_i ($i = 1, 2, \dots, s$) has $n(v_i) = n_i$ with a suitable indexing of the sinks.*

2 Truncated Trees

For a finite dimensional Leavitt path algebra $L_K(E)$ of a graph E , we would like to construct a distinguished graph F having the Leavitt path algebra isomorphic to $L_K(E)$ as follows:

Theorem 2 *Let E be a finite, acyclic graph with no isolated points. Let $s = |S(E)|$ where $S(E)$ is the set of sinks of E and $N = \max\{n(v) \mid v \in S(E)\}$. Then there exists a unique (up to isomorphism) tree F with exactly one source and $s + N - 1$ vertices such that $L_K(E) \cong L_K(F)$.*

Proof. Let the sinks v_1, v_2, \dots, v_s of E be indexed such that

$$2 \leq n(v_1) \leq n(v_2) \leq \dots \leq n(v_s) = N.$$

Define a graph $F = (F^0, F^1, r, s)$ as follows:

$$\begin{aligned} F^0 &= \{u_1, u_2, \dots, u_N, w_1, w_2, \dots, w_{s-1}\} \\ F^1 &= \{e_1, e_2, \dots, e_{N-1}, f_1, f_2, \dots, f_{s-1}\} \\ s(e_i) &= u_i \quad \text{and} \quad r(e_i) = u_{i+1} \quad i = 1, \dots, N-1 \\ s(f_i) &= u_{n(v_i)-1} \quad \text{and} \quad r(f_i) = w_i \quad i = 1, \dots, s-1. \end{aligned}$$

Clearly, F is a directed tree with unique source u_1 and $s+N-1$ vertices. F has exactly s sinks, namely $u_N, w_1, w_2, \dots, w_{s-1}$ with $n(u_N) = N$, $n(w_i) = n(v_i)$, $i = 1, \dots, s-1$. Therefore, $L_K(E) \cong L_K(F)$.

For the uniqueness part, take a tree T with exactly one source and $s+N-1$ vertices such that $L_K(E) \cong L_K(T)$. Since $N = \max\{n(v) \mid v \in S(E)\}$ which is equal to the square root of the maximum of the K -dimensions of the minimal ideals of $L_K(E)$ and hence $L_K(T)$, there exists a sink v in T with $|\{\mu_i \in T^* \mid r(\mu_i) = v\}| = N$. On the other hand, since T is a tree with a unique source and hence any vertex is connected to the unique source by a uniquely determined path, we see that the unique path joining v to the source must contain exactly N vertices, say a_1, \dots, a_{N-1}, v where a_1 is the unique source and the length of the path joining a_k to a_1 being equal to $k-1$ for any $k = 1, 2, \dots, N-1$. As $L_K(E) = \bigoplus_{i=1}^s M_{n_i}(K)$ with s summands, the remaining $s-1$ vertices must then all be sinks by Proposition 1 (4), say b_1, \dots, b_{s-1} . Since for any vertex a different from the unique source we have $n(a) > 1$ we see that for each $i = 1, \dots, s-1$ there exists an edge g_i with $r(g_i) = b_i$. Since $s(g_i)$ is not a sink we see that $s(g_i) \in \{a_1, a_2, \dots, a_{N-1}\}$, more precisely $s(g_i) = a_{n(b_i)-1}$, $i = 1, 2, \dots, s-1$. Thus T is isomorphic to F . ■

Observe that the F constructed in Theorem 2 is the tree with one source and smallest possible number of vertices ($s+N-1$) having $L_K(F)$ isomorphic to $L_K(E)$. We call F constructed in Theorem 2 as the *truncated tree associated with E* .

Proposition 3 *With the above definition of F , there is no tree T with $|T^0| < |F^0|$ such that $L_K(T) \cong L_K(F)$.*

Proof. Notice that since T is a tree, any vertex contributing to a sink represents a unique path ending at that sink. Assume on the contrary there exists a tree T with n vertices and $L_K(T) \cong A = \bigoplus_{i=1}^s M_{n_i}(K)$ such that $n < s+N-1$. Since N is the maximum of n_i 's there exists a sink with N vertices contributing. But in T the number $n-s$ of vertices which are not sinks is less than $N-1$. Hence the maximum contribution to any sink can be at most $n-s+1$ which is strictly less than N . This is the desired contradiction. ■

However if we omit the tree assumption then it is possible to find a graph G with smaller number of vertices having $L_K(G)$ isomorphic to $L_K(E)$ as the next example illustrates.

Example 4

Both $L_K(G) \cong M_3(K) \cong L_K(F)$ and $|G^0| = 2$ where as $|F^0| = 3$.

Given F_1, F_2 truncated trees associated with graphs G_1 and G_2 respectively, then $F_1 \cong F_2$ iff $L_K(F_1) \cong L_K(F_2)$ so there is a one-to-one correspondence between the Leavitt path algebra and truncated trees.

For a given finite dimensional Leavitt path algebra $A = \bigoplus_{i=1}^s M_{n_i}(K)$ with $2 \leq n_1 \leq n_2 \leq \dots \leq n_s = N$, the number s is the number of minimal ideals of A and N^2 is the maximum of the dimensions of these ideals. Therefore $\kappa(A) = s + N - 1$ is a uniquely determined algebraic invariant of A . Given $m \geq 2$, the number of isomorphism classes of finite dimensional Leavitt path algebras A which do not have any ideals isomorphic to K and $\kappa(A) = m$ is equal to the number of distinct truncated trees with m vertices by the previous paragraph. The next proposition computes this number.

Definition 5 Define a function $d : E^0 \rightarrow \mathbb{N}$ such that for any $u \in E^0$,

$$d(u) = |\{v \mid n(v) \leq n(u)\}|.$$

Observe that in a truncated tree, the restriction of the function d on the set of vertices which are not sinks is one to one.

Proposition 6 The number of distinct truncated trees with n vertices is 2^{n-2} .

Proof. For every truncated tree E with n vertices we assign an n -vector $\alpha(E) = (\alpha_1, \alpha_2, \dots, \alpha_n)$ where $\alpha_i \in \{0, 1\}$ as follows:

- $\alpha(E)$ contains exactly $N - 1$ many 1's where $N - 1$ is the number of non-sinks of E .
- To define that vector it is sufficient to know which component is 1.
- To each vertex v which is not a sink, we assign a 1 appearing in the $d(v)$ -th component.
- Remaining components are all zero.

Hence $\alpha(E)$ starts with 1 and ends with 0.

Given any $\{0, 1\}$ sequence β of length n starting with 1 and ending with 0, there exists clearly a unique truncated tree E with n vertices such that $\alpha(E) = \beta$. Hence the number of distinct truncated trees with n vertices is equal to the number of all $\{0, 1\}$ -sequences of length n in which the first and last components are constant which is equal to 2^{n-2} . ■

For a tree F with n vertices the K -dimension of $L_K(F)$ is not uniquely determined by the number of vertices only. However, we can compute the maximum and the minimum K -dimensions of $L_K(F)$ where F ranges over all possible trees with n vertices.

Lemma 7 The maximum K -dimension of $L_K(E)$ where E ranges over all possible trees with n vertices and s sinks is equal to $s(n - s + 1)^2$.

Proof. Assume E is a tree with n vertices. Then $L_K(E) \cong \bigoplus_{i=1}^s M_{n_i}(K)$, by Proposition 1 (3) where s is the number of sinks in E and $n_i \leq n - s + 1$ for all $i = 1, \dots, s$. Hence

$$\dim L_K(E) = \sum_{i=1}^s n_i^2 \leq s(n - s + 1)^2.$$

Notice that there exists a tree E as sketched below

with n vertices and s sinks such that $\dim L_K(E) = s(n - s + 1)^2$. ■

Theorem 8 *The maximum K -dimension of $L_K(E)$ where E ranges over all possible trees with n vertices is given by $f(n)$ where*

$$f(n) = \begin{cases} \frac{n(2n+3)^2}{27} & \text{if } n \equiv 0 \pmod{3} \\ \frac{1}{27}(n+2)(2n+1)^2 & \text{if } n \equiv 1 \pmod{3} \\ \frac{4}{27}(n+1)^3 & \text{if } n \equiv 2 \pmod{3} \end{cases}$$

Proof. Assume E is a tree with n vertices. Then $L_K(E) \cong \bigoplus_{i=1}^s M_{n_i}$ where s is the number of sinks in E . Now, to find $\max \dim L_K(E)$ we need only to determine maximum value of the function $f(s) = s(n - s + 1)^2$ for $s = 1, 2, \dots, n - 1$. Extending the domain of $f(s)$ to real numbers $1 \leq s \leq n - 1$ we get a continuous function, hence we can find its maximum value.

$$f(s) = s(n - s + 1)^2 \Rightarrow \frac{d}{ds} (s(n - s + 1)^2) = (n - 3s + 1)(n - s + 1)$$

Then $s = \frac{n+1}{3}$ is the only critical point in the interval $[1, n - 1]$ and since $\frac{d^2 f}{ds^2}(\frac{n+1}{3}) < 0$, it is a local maximum. In particular f is increasing on $\left[1, \frac{n+1}{3}\right]$ and decreasing on $\left[\frac{n+1}{3}, n - 1\right]$. We have three cases:

Case 1: $n \equiv 2 \pmod{3}$. In this case $s = \frac{n+1}{3}$ is an integer and maximum K -dimension of $L_K(E)$ is $f\left(\frac{n+1}{3}\right) = \frac{4}{27}(n+1)^3$ and we have $n_i = \frac{2(n+1)}{3}$, for each $i = 1, 2, \dots, s$.

Case 2: $n \equiv 0 \pmod{3}$. Then we have: $\frac{n}{3} = t < t + \frac{1}{3} = s < t + 1$ and

$$f\left(\frac{n}{3}\right) = \frac{(2n+3)^2 n}{27} = \alpha_1 \text{ and } f\left(\frac{n}{3} + 1\right) = \frac{4n^2(n+3)}{27} = \alpha_2.$$

Note that, $\alpha_1 > \alpha_2$. So α_1 is maximum K -dimension of $L_K(E)$ and we have $n_i = \frac{2}{3}n + 1$, for each $i = 1, 2, \dots, s$.

Case 3: $n \equiv 1 \pmod{3}$. Then $\frac{n-1}{3} = t < t + \frac{2}{3} = s < t + 1$ and

$$f\left(\frac{n-1}{3}\right) = \frac{4}{27}(n+2)^2(n-1) = \beta_1$$

and

$$f\left(\frac{n+2}{3}\right) = \frac{1}{27}(2n+1)^2(n+2) = \beta_2.$$

In this case $\beta_2 > \beta_1$ and so β_2 gives the maximum K -dimension of $L_K(E)$ and we have $n_i = \frac{2n+1}{3}$, for each $i = 1, 2, \dots, s$. ■

Theorem 9 *The minimum K -dimension of $L_K(E)$ where E ranges over all possible trees with n vertices and s sinks is equal to $r(q+2)^2 + (s-r)(q+1)^2$, where $n-1 = qs+r$, $0 \leq r < s$.*

Proof. We call a graph a *bunch tree* if it is obtained by identifying the unique sources of the finitely many oriented finite line graphs.

Let $\mathcal{E}(n, s)$ be the set of all bunch trees with n vertices and s sinks.

Every element of $\mathcal{E}(n, s)$ can be uniquely represented by an s -tuple (t_1, t_2, \dots, t_s) where each t_i is the number of vertices contributing only to the i -th sink with $1 \leq t_1 \leq t_2 \leq \dots \leq t_s$ and $t_1 + t_2 + \dots + t_s = n - 1$.

Let $E \in \mathcal{E}(n, s)$ with $t_s - t_1 \leq 1$. This E is represented by the s -tuple $(q, \dots, q, q+1, \dots, q+1)$ where $n-1 = sq+r$, $0 \leq r < s$.

Now we claim that the dimension of E is the minimum of the set

$$\{\dim L_K(F) : F \text{ tree with } s \text{ sinks and } n \text{ vertices}\}.$$

If we represent $U \in \mathcal{E}(n, s)$ by the s -tuple (u_1, u_2, \dots, u_s) then $E \neq U$ implies that $u_s - u_1 \geq 2$.

Consider the s -tuple (t_1, t_2, \dots, t_s) where (t_1, t_2, \dots, t_s) is obtained from $(u_1 + 1, u_2, \dots, u_{s-1}, u_s - 1)$ by reordering the components in increasing order.

In this case the dimension d_U of U is

$$d_U = (u_1 + 1)^2 + \dots + (u_s + 1)^2.$$

Similarly, the dimension d_T of the bunch graph T represented by the s -tuple (t_1, t_2, \dots, t_s) , is

$$d_T = (t_1 + 1)^2 + \dots + (t_s + 1)^2 = (u_1 + 2)^2 + \dots + u_{s-1}^2 + u_s^2.$$

Hence

$$d_U - d_T = 2(u_s - u_1) - 2 > 0.$$

Repeating this process sufficiently many times we see that the process has to end at the exceptional bunch tree E showing that its dimension is the smallest among the dimensions of all elements of $\mathcal{E}(n, s)$.

Now let F be an arbitrary tree with n vertices and s sinks. As above we assign to F the s -tuple (n_1, n_2, \dots, n_s) with $n_i = n(v_i) - 1$ where the sinks v_i , $i = 1, 2, \dots, s$ are indexed in such a way that $n_i \leq n_{i+1}$, $i = 1, \dots, s-1$.

Observe that $n_1 + n_2 + \dots + n_s \geq n-1$. Let $\beta = \sum_{i=1}^s n_i - (n-1)$. Since $s \leq n-1$, $\beta \leq \sum_{i=1}^s (n_i - 1)$. Either $n_1 - 1 \geq \beta$ or there exists a unique $k \in \{2, \dots, s\}$ such that $\sum_{i=1}^{k-1} (n_i - 1) < \beta \leq \sum_{i=1}^k (n_i - 1)$. If $n_1 - 1 \geq \beta$, then let

$$m_i = \begin{cases} n_1 - \beta & , \quad i = 1 \\ n_i & , \quad i > 1 \end{cases}.$$

Otherwise, let

$$m_i = \begin{cases} 1 & , \quad i \leq k-1 \\ n_k - \left(\beta - \sum_{i=1}^{k-1} (n_i - 1) \right) & , \quad i = k \\ n_i & , \quad i \geq k+1 \end{cases}.$$

In both cases, the s -tuple (m_1, m_2, \dots, m_s) that satisfies $1 \leq m_i \leq n_i$, $m_1 \leq m_2 \leq \dots \leq m_s$ and $m_1 + m_2 + \dots + m_s = n-1$ is obtained. So, there exists a bunch tree M namely the one corresponding uniquely to (m_1, m_2, \dots, m_s) which has dimension $d_M \leq d_F$. This implies that $d_F \geq d_E$.

Hence the result follows. ■

Lemma 10 *The minimum K -dimension of $L_K(E)$ where E ranges over all possible trees with n vertices occurs when the number of sinks is $n-1$ and is equal to $4(n-1)$.*

Proof. By the previous theorem we see that

$$\dim L_K(E) \geq r(q+2)^2 + (s-r)(q+1)^2$$

where $n-1 = qs+r$, $0 \leq r < s$. We have

$$r(q+2)^2 + (s-r)(q+1)^2 = (n-1)(q+2) + qr + r + s.$$

Thus

$$(n-1)(q+2) + qr + r + s - 4(n-1) = (n-1)(q-2) + qr + r + s \geq 0 \text{ if } q \geq 2.$$

If $q = 1$, then $-(n-1) + 2r + s = -(n-1) + r + (n-1) = r \geq 0$. Hence $\dim L_K(E) \geq 4(n-1)$.

Notice that there exists a truncated tree E with n vertices and $\dim L_K(E) = 4(n-1)$ as sketched below :

■

3 Line Graphs

The *total-degree* of the vertex v is the number of edges that either have v as its source or as its range, that is, $\text{tot deg}(v) = |s^{-1}(v) \cup r^{-1}(v)|$. A finite graph E is a *line graph* if it is connected, acyclic and $\text{totdeg}(v) \leq 2$ for every $v \in E^0$.

Remark 11 In [1], the proposition 5.7 shows that a semisimple finite dimensional algebra $A = \bigoplus_{i=1}^s M_{n_i}(K)$ over the field K can be described as a Leavitt path algebra $L(E)$ defined by a line graph E , if and only if A has no ideals of K -dimension 1 and the number of minimal ideals of A of K dimension 2^2 is at most 2. On the other hand, if $A \cong L(E)$ for some n line graph E then $n-1 = \sum_{i=1}^s (n_i - 1)$, that is, n is an algebraic invariant of A .

Therefore the following proposition answers a reasonable question.

Proposition 12 The number A_n of isomorphism classes of Leavitt path algebras defined by line graphs having exactly n vertices is

$$A_n = P(n-1) - P(n-4)$$

where $P(m)$ is the number of partitions of the natural number m .

Proof. Any n -line graph has $n - 1$ edges. In a line graph, for any edge e there exists a unique sink v so that there exists a path from $s(e)$ to v . In this case we say that e is directed towards v . The number of edges directed towards v is clearly equal to $n(v) - 1$. Let E and F be two n -line graphs. $L_K(E) \cong L_K(F)$ if and only if there exists a bijection $\phi : S(E) \rightarrow S(F)$ such that for each v in $S(E)$, we have $n(v) = n(\phi(v))$. Therefore the number of isomorphism classes of Leavitt path algebras determined by n -line graphs is the number of partitions of $n - 1$ edges in which the number of parts having exactly one edge is at most two. Since the number of partitions of k objects having at least three parts each of which containing exactly one element is $P(k - 3)$, we get the result $A_n = P(n - 1) - P(n - 4)$. ■

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